Solutions

Exam 2
Chapter 3 and 4.1-4.5

Short Answer
Answer the following questions. *You must show your work to receive full credit.* Be sure to make reasonable simplifications. Indicate your final answer with a $\boxed{\text{box}}$.

1. Find the derivative of the given function.

(a) (4 points) $f(t) = \sin(t^2 - 1)$.

(b) (6 points) $g(x) = \left(\frac{x^3}{\ln x}\right)^2$.

(a) $f'(t) = 2t \cdot \cos(t^2 - 1)$

(b) $g'(x) = 2 \cdot \left(\frac{x^3}{\ln x}\right) \cdot \frac{\ln x \cdot 3x^2 - x^3 \cdot \frac{1}{x}}{(\ln x)^2}$

$= \frac{2x^5(3\ln x - 1)}{(\ln x)^3}$.

2. (10 points) The position at time $t \geq 0$ of a particle moving along a horizontal coordinate line is given by

$$s = 20\cos(t + \pi/2).$$

Find the particle's starting position, furthest distance left and right, and its velocity,speed and acceleration.

Starting Position: -20 at $t = \pi/2$ $\quad 0$ at $t=0$ $\quad 20$ at $t = 3\pi/2$

$$s(0) = 20 \cdot \cos(0 + \pi/2) = 0$$

Furthest Distance Left (Right):

$\max(\cos(t + \pi/2)) = 1$ at $t = \frac{3\pi}{2} \Rightarrow \text{Right} = 20$

$\min(\cos(t + \pi/2)) = -1$ at $t = \pi/2 \Rightarrow \text{Left} = -20$

$\text{Velocity} \equiv v(t) = s'(t) = 20 \cdot (-\sin(t + \pi/2)) = -20\sin(t+\pi/2)$

$\text{Speed} \equiv |v(t)| = 20\,|\sin(t + \pi/2)|$

$\text{Acceleration} \equiv a(t) = v'(t) = -20\cos(t + \pi/2).$

3. The demand equation for a quantity q of a product at price p, in dollars, is $p = -5q + 5000$. Companies producing the product report the cost, C, in dollars, to produce a quantity q is $C = 10q + 5$.

(a) (2 points) Find the revenue function R as function of quantity, q. (Hint: Revenue$=R = pq$.)

(b) (2 points) Find the profit function P. (Hint: Profit$=P = R - C$)

(c) (6 points) For what quantity q will profit be maximized? What is the maximum profit?

(a) Revenue $\equiv R = p \cdot q = (-5q + 5000) \cdot q = -5q^2 + 5000q$. dollars

(b) Profit $\equiv P = R - C = -5q^2 + 5000q - (10q + 5)$

$= -5q^2 + 4990q - 5$. dollars

(c) $P'(q) = -10q + 4990$

Crit. Pt at $q = 499$.

$P''(q) = -10$ so always concave down and thus $q = 499$ corresponds to local max.

Thus max profit is at $q = 499$ and is $P(499) = -5(499)^2 + 4990 \cdot 499 - 5$ dollars.

4. (10 points) Consider the function $f(x) = x^2 + \frac{2}{x} = \frac{x^3+2}{x}$. Find all critical and inflection points. Identify the behavior of f on the interval and which critical points correspond to local extrema. For a bonus 5 points identify the asymptotes of f and graph it.

$\mathrm{Dom}(f) = \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$

$f'(x) = 2x - \frac{2}{x^2} = 2\left(\frac{x^3 - 1}{x^2}\right)$ Critical point at $x = 1$

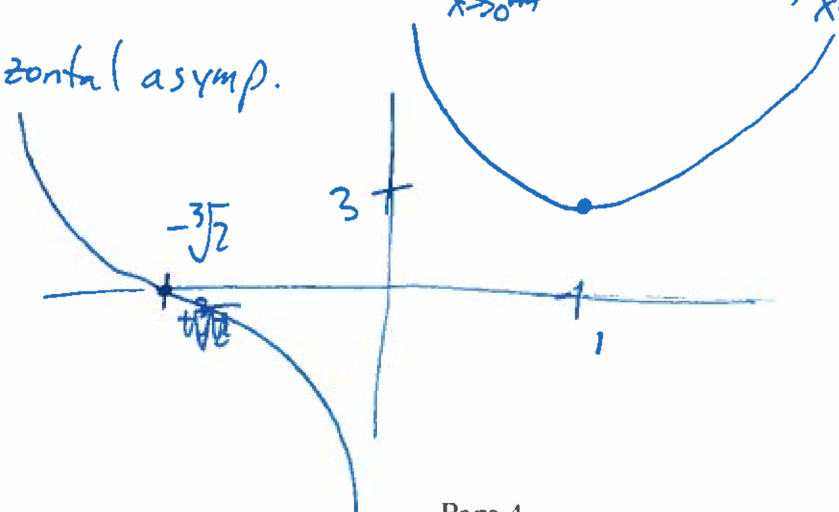
$f''(x) = 2 + 2 \cdot \frac{2}{x^3} = 2\left(\frac{x^3+2}{x^3}\right)$ Possible inflection point at $x = -\sqrt[3]{2}$.

f	dec/con↑	dec/con↓	dec/con↑	inc/con↑
f'	$- \; -$	$- \; -$	$- \; -$	$+ \; +$
f''	$+ \; +$	$- \; -$	$+ \; +$	$+ \; +$
	$-\sqrt[3]{2}$	0	1	

So $x = -\sqrt[3]{2}$ is an inflection point and $x = 1$ is a local min.

Bonus: $x = 0$ is a vertical asymp. $\lim_{x \to 0^+} f(x) = \infty$, $\lim_{x \to 0^-} f(x) = -\infty$

No horizontal asymp.



5. Find the desired limits.

(a) (5 points) $\lim_{x \to 0} \frac{1-\cos x}{x^2}$.

(b) (5 points) $\lim_{x \to 0^+} \left(1 + \frac{1}{x}\right)^x$.

(a) $$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \frac{0}{0}$$

$\Big\|$ L'Hopital

$$\lim_{x \to 0} \frac{\sin x}{2x} = \frac{0}{0}$$

$\Big\|$ L'Hopital

$$\lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}$$

(b) $$\lim_{x \to 0^+} \left(1+\frac{1}{x}\right)^x = \infty^0$$

Take logs: $$\lim_{x \to 0^+} \ln\left(\left(1+\frac{1}{x}\right)^x\right) = \lim_{x \to 0^+} x \cdot \ln\left(1+\frac{1}{x}\right) = 0 \cdot \infty$$

Rewrite: $$\lim_{x \to 0^+} \frac{\ln\left(1+\frac{1}{x}\right)}{1/x} = \frac{\infty}{\infty}$$

$\Big\|$ L'Hopital

$$\lim_{x \to 0^+} \frac{\frac{1}{1+\frac{1}{x}} \cdot \frac{-1}{x^2}}{-1/x^2} = \lim_{x \to 0^+} \frac{1}{1+\frac{1}{x}} = 0.$$

Bonus Question: (10 points) Allisha Gray is getting some Gatorade after a hard fought victory in the NCAA tournament. Assuming that the Gatorade cooler is a right circular cylinder, the volume V of the liquid remaining in the cooler is given by

$$V = \pi r^2 h \ \text{ in}^3$$

where h is the height and r is the radius. Furthermore, assume that the radius is a constant of 8 inches. Find dV/dt if the height is decreasing at a constant rate of $\frac{0.1}{\pi}$ inches per second.

$$\frac{dV}{dt} = \pi r^2 \cdot \frac{dh}{dt} \ \text{in}^3/\text{sec.}$$

Evaluate at $r = 8$ in. and $\frac{dh}{dt} = \frac{0.1}{\pi}$ in/sec

to find

$$\frac{dV}{dt} = \frac{6.4}{\pi} \cdot \pi \ \text{in}^3/\text{sec} = 6.4 \ \text{in}^3/\text{sec.}$$